Langlands Conjectures

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June 27, 2024

This is largely based on [BCDS+04], [Sha10], [Bor] and [Lan].

1 Dual Groups

If k is a number field we know that there is an equivilence of categories between connected reductive groups defined over $\bar{k}$ and the category of root datum, with morphisms as isogenies. Thus to any root datum we can associate a reductive group defined over $\bar{k}$. Now if we start with a connected reductive group G defined over a number field k, then we can associate a root datum, by base changing to $\bar{k}$. Therefore we have a way of associating to each reductive group over a number field a reductive group defined over the algebraic closure.

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$$G/k \mapsto \hat{G}/\bar{k}$$

This we call the dual group. We can also capture the difference in the two fields by looking at the Galois group $Gal(\bar{k}/k)$ and this leads us to consider the so called L-group associated to G

$$G \mapsto {}^L G \ := \ \hat{G} \rtimes Gal(\bar{k}/k)$$

Note that if G is split over k then this is a direct product. We should also note that $Gal(\bar{k}/k)$ acts on $\hat{G}$ via condsidering a splitting of an exact sequence of the automorphism groups and the group cohomology, details in [Sha10].

Is this a reductive group? Defined over what? Check this is right...

2 Satake Isomorphism

For a finite place ν of k Langlands interpretation of the Satake isomorphism allows you to associate to certain representations of $G(k_\nu)$ conjugacy classes of the dual group ${}^LG$. Hence we can "plug in" representations of $G(k_\nu)$ into representations of ${}^LG$. We ellaborate further.

2.1 A Lattice

We consider a maximal torus $T \subseteq G$ defined over k_ν. For simplification [Sha10] restricts to G being quasi-split, we follow. This allows one to take T to further be a maximally split maximal torus. Notice: This is terrible language. What is meant is that T has maximal split rank and is moreover a maximal torus. This is important becuase T might not be split (it is only G_m^n after base change). But having maximal split rank ensures that it contains the maximal split torus of G (the split rank is the rank of the maximal split torus that you contain). We call the maximal split torus $A \subseteq T$, which becuase T is maximally split is also a maximal split torus of G. We have the important map

$$H_T : T(k_\nu) \to X_*(T)_{k_\nu} \quad := \quad \mathrm{Hom}(X^*(T)^{Gal(\bar{k_\nu}/k_\nu)}, \mathbb{Z})$$

$$H_T(t)(\chi) = \log_q |\chi(t)|_{k_\nu}$$

where as usual $X^*(T)$ is the module of characters of T, and on the left is the Galois fixed points of those characters. We denote the image of this function by $\Lambda \quad := \quad H_T(T)$.

what is q, Shahidi doesnt say what the base is. It shouldnt really matter but still...

Notice we have an action of the Weyl group $W = W(A, G)$ on Λ, this naturally extends to the algebra $\mathbb{C}[\Lambda]$.

2.2 Ramification

G is said to be unramified over k_ν if it is quasi-split to split over an unramified (cyclic) field extension of k_ν. A character $\chi : T(k_\nu) \to \mathbb{C}^*$ is unramified if it factors through Λ. An irreducible admissible complex represntation (π, V) of $G(k_\nu)$, where G is unramified over k_ν, is said to be unramified if V has a fixed vector under the action of $G(\mathcal{O}_{k_\nu})$.

2.3 Hecke Algebra

If G is unramified then $K \quad := \quad G(\mathcal{O}_{k_\nu})$ is a maximal compact open subgroup of $G(k_\nu)$, then we define the algebra

$$\mathcal{H}(G(k_\nu), K)$$

to be the K-bi-invariant smooth functions of compact support on $G(k_\nu)$, with multiplication given by convolution.

2.4 Two Isomorphisms

If we assume that G is unramified again then in fact we have the isomorphism of $\mathbb{C}$ algebras

$$\mathcal{H}(T(k_\nu), T(k_\nu) \cap K) \cong \mathbb{C}[\Lambda]$$

there is also the Satake transformation

$$S : \mathcal{H}(G(k_\nu), K) \to \mathcal{H}(T(k_\nu), T(k_\nu) \cap K)$$

$$S(f)(t) \quad := \quad \delta(t)^{\frac{1}{2}} \int_U f(tu)du$$

The details will not be important here. The Satake isomorphism then composes these and we get that S is an isomorphism onto the subspace

$$\mathcal{H}(G(k_\nu), K) \to \mathbb{C}[\Lambda]^W$$

a corrolary of the Satake isomorphism is that $\mathrm{Hom}(\mathcal{H}(G(k_\nu), K), \mathbb{C}) \cong X_{un}/W$ where X_{un} is the collection of unramified characters of $T(k_\nu)$. This is explicitly given by the association of a character χ of $T(k_\nu)$ to the homomorphism

$$\omega_\chi : \mathcal{H}(G(k_\nu), K) \to \mathbb{C}$$

$$\omega_\chi(f) \quad := \quad \int_{T(k_\nu)} S(f)(\chi(t))dt$$

The second isomorphism we need to understand is

$$\mathbb{C}[\hat{A}]^W \cong \mathfrak{A}$$

where $\mathfrak{A}$ is the algebra of restrictions of elements of $Rep({}^LG)$ to $(\hat{G} \rtimes \sigma)_{ss}/Int\ \hat{G}$, where σ is the Frobenius element of $Gal(\bar{k}_\nu/k_\nu)$ and the ss subscript denotes the subset of semi-simple elements, Int is the internal automorphisms of $\hat{G}$, i.e. automorphisms given by conjugations. there are more isomorphisms to understand but I think that its not too important at the moment.

2.5 Interpretation

Let G, T, A etc be as above. Now let π be an irreducible unramified representation of $G(k_\nu)$. Via the isomorphisms above we get an associated element of $c(\pi) \in (\hat{G} \rtimes \sigma)_{ss}/Int\ \hat{G}$ determined up to conjugation of π. i.e. the conjugacy class of π determines a unique conjugacy class in $(\hat{G} \rtimes \sigma)_{ss}$.

3 L Functions

3.1 Local

Now we can define our L-functions. Note that we are still requiring G to be unramified over k_ν. We need three parameters, a complex number s, an irreducible unramified representation π of $G(k_\nu)$ and a finite dimensional complex representation of ${}^LG$, r. Then the local L-function is

$$L(s, \pi, r) \quad := \quad \det(I - r(c(\pi))q^{-s})^{-1}$$

3.2 Global

A little bit of work needs to be done. First we need the fact that G is unramified over all but finitely many k_ν, i.e. ramifies at only finitely many places. Next we use the fact that representations of $G(\mathbb{A})$ are representatiosn of $\mathcal{H}(G(\mathbb{A}), K)$ (for the definition of this Hecke algebra see the references in [Sha10], it is more subtle in the infinite places) and then decompose this algebra into the tensor product of local Hecke algebras $\mathcal{H}(G(k_\nu), K_\nu)$. Now given an irreducible admissible representatino of $G(\mathbb{A})$, say π, its restriction to the K finite vectors decomposes into a (restricted) tensor product over all places,

$$\pi|_{K-fin} = \otimes_\nu \pi_\nu$$

such that the π_ν are irreducible admissible reps of the local Hecke algebras, that are moreover unramified for almost all ν. This is why for the definition of L-functions we consider only representations that are tensor products over local representations.

Am I right here? or do reps of $G\mathbb{A}$ always look like tensors of local reps?

Finally let $\pi = \otimes_\nu \pi_\nu$ be an irreducible admissible unitary representation of $G(\mathbb{A})$ such that for all $v \notin S$, a finite set of places, both G and π_ν are unramified. Let r be a finite dimensional complex representation of ${}^LG$ then we define the partial global L-function as

$$L^S(s,\pi,r) \quad := \quad \prod_{\nu \notin S} L(s,\pi_\nu,r_\nu)$$

where r_ν is the induced complex representation of ${}^LG_{k_\nu}$, i.e. the base change of the L group to the local field, induced from the inclusion of the Galois groups

I dont understand this, isnt base change functorial, I feel like I should be able to just apply it to the rep wholesale?

$$Gal(\bar{k}_\nu/k_\nu) \hookrightarrow Gal(\bar{k}/k)$$

4 The Conjectures

Langlands made two types of conjectures. The first is that one can define certain L-functions and that they would have certain nice properties. This can be seen as a generalisation of what we know (and want to know) about the Riemann zeta function. Moreover these fundamental properties of the L-functions are prerequisites for the second conjectures, both to be well posed and tractable. The second conjecture type is akin to class field theory and the modularity conjecture/theorem positing some relation between the theory of automorphic forms and the L-functions.

4.1 About L-Functions

The original conjectures as posed by Langlands in his letter to Weil, and latter expanded apon in [Lan] ask simple questions about these L-functions.

- Are they meromorphic on all of $\mathbb{C}$
- Do they satisfy some nice functional equations
- Do we have control over their poles (simple and finite)

This was his first question. His second question was whether it was possible to view the conjugacy classes of $(\hat{G} \rtimes \sigma)_{ss}$ as comming from automorphic forms.

The mythos has expanded and changed as time went on, some of the original ideas needed to be tweaked (they were not true but slight variations are, see packets) and more specifics were added to make the problems well posed. A part of the global story is the mere definition of global L-functions. That is we have only defined the local L-functions at unramified places, we would like to have a definition at all places such that the global L-function would be the product over all of them and would satisfy some properties (so as to not make it trivial). This is still open, however has apparently been fully worked out for Gl_n, and this informs the requirments of the general statement.

4.2 About Functoriality

First we need the notion of an L-homomorphism. Let G, H be two connected reductive groups defined over k. Then a homomorphism of their L-Groups $u : {}^LH \to {}^LG$ is one that respects the relevant structures. In particular we have that

- It preserves the Galois strucutre, the following diagram commutes

$$\begin{array}{ccc} {}^LH & \xrightarrow{\quad u \quad} & {}^LG \\ \downarrow & & \downarrow \\ Gal(\bar{k}/k) & = & Gal(\bar{k}/k) \end{array}$$

- It preserves the topology, u is continuous
- It prserves the complex group, u restricts to a complex analytic homomorphism

$$u : \hat{H} \to \hat{G}$$

then the conjecture is as follows: If k is a local / global and G, H are connected reductive k-groups such that G is quasi-split then we want to associate to every L-homomorphism $u : {}^L H \to {}^L G$ a lift of admissible / autommorphic representations of H to admissible / autommorphic representations of G.

$$\begin{array}{ccc} {}^L H & \xrightarrow{\quad u \quad} & {}^L G \\ & \Big\Downarrow L & \\ \mathrm{Rep}_A(H) & \xrightarrow{\quad L(u) \quad} & \mathrm{Rep}_A(G) \end{array}$$

Or alternitively it is the postulation of a functor

L-Groups, L-Homomorphisms $\to$ Categories of Representations

or perhaps a family of functors that depend on the field?

Moreover one wants this functor to preserve the L-functions we defined and satisfy whatever else.

5 Interpretations

There are other statements that are stated as the Langlands conjectures and we ellaborate here on where they come from. In particular we dwell on some particular special cases of functoriality that are of special intrest.

5.1 Non-Abelian Class Field Theory

The generalisation of class field theory can be seen by considering the Langlands dual of the trivial group. It is clear that

$${}^L\{1\} = Gal(\bar{k}/k)$$

hence the conjectures that are often stated as the Langlands conjectures, the correspondence between representations of some reductive groups and certain types of Galois representations can be recovered from the notion of functoriality expressed above.

5.2 More

There are many other things that are said to be implied by the Langlands correspondence although I admit I dont understand them. I would like to maintain this document and come back to explain these at some point:

- Modularity conjecture
- Artin reciprocity (mentioned by Langlands in the original letter)

References

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